JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON WEDNESDAY 29th JANUARY 2025) **TIME : 3:00 PM TO 6:00 PM**

MATHEMATICS **TEST PAPER WITH SOLUTION**

SECTION-A

1. If the set of all a ∈ **R**, for which the equation $2x^2 + (a-5)x + 15 = 3a$ has no real root, is the interval (α,β), and $X = \{x \in Z : \alpha < x < \beta\}$, then $\sum_{x\in X} x^2$ is equal to

(1) 2109 (2) 2129

(3) 2139 (4) 2119

Ans. (3)

Sol. $(a-5)^2 - 8(15-3a) < 0$

$a^2 + 14a + 25 - 120 < 0$

$a^2 + 14a - 95 < 0$

$(a+19)(a-5) < 0$

$a \in (-19, 5)$

$\therefore -19 < x < 5$

$\therefore \sum_{x\in X} x^2 = (1^2 + 2^2 + \ldots. + 4^2) + (1^2 + 2^2 + \ldots + 18^2)$

$= \dfrac{4\times5\times9}{6} + \dfrac{18\times19\times37}{6}$

$= 30 + 2109$

$= 2139$

2. If $\sin x + \sin^2 x = 1,\ x \in \left(0, \dfrac{\pi}{2}\right)$, then $(\cos^{12}x + \tan^{12}x) + 3(\cos^{10}x + \tan^{10}x + \cos^8 x + \tan^8 x) + (\cos^6 x + \tan^6 x)$ is equal to

(1) 4 (2) 3

(3) 2 (4) 1

Ans. (3)

Sol. $\sin x + \sin^2 x = 1$

$\Rightarrow \sin x = \cos^2 x \Rightarrow \tan x = \cos x$

$\therefore$ Given expression

$= 2\cos^{12}x + 6[\cos^{10}x + \cos^8 x] + 2\cos^6 x$

$= 2[\sin^6 x + 3\sin^5 x + 3\sin^4 x + \sin^3 x]$

$= 2\sin^3 x[(\sin x + 1)^3]$

$= 2[\sin^2 x + \sin x]^3$

$= 2$

3. Let the area enclosed between the curves $|y| = 1 - x^2$ and $x^2 + y^2 = 1$ be α. If $9\alpha = \beta\pi + \gamma$; β, γ are integers, then the value of $|\beta - \gamma|$ equals

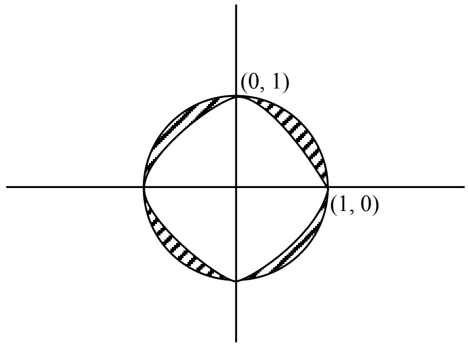
(1) 27 (2) 18

(3) 15 (4) 33

Ans. (4)

Sol. $C_1 : |y| = 1 - x^2$

$C_2 : x^2 + y^2 = 1$



$\therefore$ Required Area

$= \alpha = 4[\text{Area of circle in } 1^{st} \text{ quad.} - \int_0^1 (1-x^2)dx\,]$

$= 4\left[\dfrac{\pi}{4} - \left[x - \dfrac{x^3}{3}\right]_0^1\right]$

$\alpha = \pi - \dfrac{8}{3}$

$\therefore 3\alpha = 3\pi - 8$

$\therefore 9\alpha = 9\pi - 24$

$\therefore \beta = 9, \gamma = -24$

$\therefore |\beta - \gamma| = 33$

4. If the domain of the function $\log_5\left(18x - x^2 - 77\right)$ is (α,β) and the domain of the function $\log_{(x-1)}\left(\dfrac{2x^2+3x-2}{x^2-3x-4}\right)$ is (γ,δ), then $\alpha^2 + \beta^2 + \gamma^2$ is equal to :

(1) 195 (2) 174

(3) 186 (4) 179

Ans. **(3)**

Sol. $f_1(x) = \log_5(18x - x^2 - 77)$

$\therefore\ 18x - x^2 - 77 > 0$

$x^2 - 18x + 77 < 0$

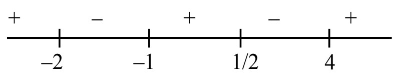
$x \in (7, 11)\quad \alpha = 7, \beta = 11$

$f_2(x) = \log_{(x-1)}\left(\dfrac{2x^2+3x-2}{x^2-3x-4}\right)$

$\therefore\ x - 1 > 0,\ x - 1 \neq 1,\ \dfrac{2x^2+3x-2}{x^2-3x-4} > 0$

$x > 1,\ x \neq 2,\ \dfrac{(2x-1)(x+2)}{(x-4)(x+1)} > 0$

$x > 1,\ x \neq 2,$



$\therefore\ x \in (4, \infty)$

$\therefore\ \gamma = 4$

$\therefore\ \alpha^2 + \beta^2 + \gamma^2 = 49 + 121 + 16$

$= 186$

5. Let the function $f(x) = (x^2 - 1)|x^2 - ax + 2| + \cos|x|$ be not differentiable at the two points $x = \alpha = 2$ and $x = \beta$. Then the distance of the point (α,β) from the line $12x + 5y + 10 = 0$ is equal to :

(1) 3 (2) 4

(3) 2 (4) 5

Ans. **Allen Ans. (BONUS)**

NTA Ans. (1)

Sol. $\cos|x|$ is always differentiable

$\therefore$ we have to check only for $|x^2 - ax + 2|$

$\therefore$ Not differentiable at

$x^2 - ax + 2 = 0$

One root is given, $\alpha = 2$

$\therefore\ 4 - 2a + 2 = 0$

$a = 3$

$\therefore$ other root $\beta = 1$

but for $x = 1$ $f(x)$ is differentiable

(Drop)

6. Let a straight line L pass through the point P(2,–1,3) and be perpendicular to the lines $\dfrac{x-1}{2} = \dfrac{y+1}{1} = \dfrac{z-3}{-2}$ and $\dfrac{x-3}{1} = \dfrac{y-2}{3} = \dfrac{z+2}{4}$.

If the line L intersects the yz-plane at the point Q, then the distance between the points P and Q is :

(1) 2 (2) $\sqrt{10}$

(3) 3 (4) $2\sqrt{3}$

Ans. **(3)**

Sol. Vector parallel to 'L'

$$= \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -2 \\ 1 & 3 & 4 \end{vmatrix} = 10\hat{i} - 10\hat{j} + 5\hat{k}$$

$= 5(2\hat{i} - 2\hat{j} + \hat{k})$

Equation of 'L'

$\dfrac{x-2}{2} = \dfrac{y+1}{-2} = \dfrac{z-3}{1} = \lambda$ (say)

Let $Q(2\lambda + 2, -2\lambda - 1, \lambda + 3)$

$\Rightarrow 2\lambda + 2 = 0 \Rightarrow \lambda = -1$

$\Rightarrow Q\ (0, 1, 2)$

$d(P, Q) = 3$

7. Let $S = \mathbf{N} \cup \{0\}$. Define a relation **R** from S to **R** by :

$$\mathbf{R} = \left\{(x,y) : \log_e y = x\log_e\left(\frac{2}{5}\right), x \in S, y \in \mathbf{R}\right\}.$$

Then, the sum of all the elements in the range of **R** is equal to

(1) $\dfrac{3}{2}$ (2) $\dfrac{5}{3}$

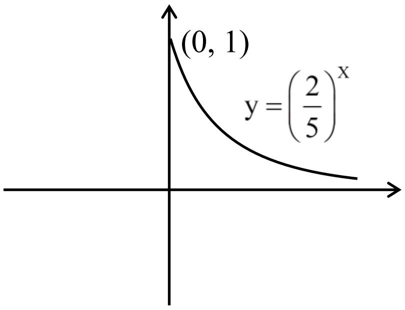
(3) $\dfrac{10}{9}$ (4) $\dfrac{5}{2}$

Ans. **(2)**

Sol. $S = \{0, 1, 2, 3 \ldots..\}$

$\log_e y = x\log_e\left(\dfrac{2}{5}\right)$

$\Rightarrow\ y = \left(\dfrac{2}{5}\right)^x$



Required

$$\text{Sum} = 1+\left(\frac{2}{5}\right)^1+\left(\frac{2}{5}\right)^2+\left(\frac{2}{5}\right)^3+.....-=\frac{1}{1-\frac{2}{5}}=\frac{5}{3}$$

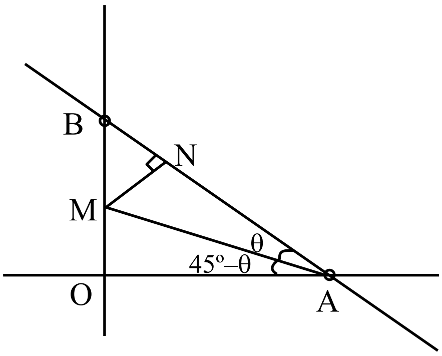
8. Let the line $x + y = 1$ meet the axes of x and y at A and B, respectively. A right angled triangle AMN is inscribed in the triangle OAB, where O is the origin and the points M and N lie on the lines OB and AB, respectively. If the area of the triangle AMN is $\frac{4}{9}$ of the area of the triangle OAB and AN : NB = λ : 1, then the sum of all possible value(s) of is λ :

(1) $\frac{1}{2}$ (2) $\frac{13}{6}$

(3) $\frac{5}{2}$ (4) 2

Ans. **(4)**

Sol.



Area of $\Delta AOB = \frac{1}{2}$

Area of $\Delta AMN = \frac{4}{9}\times\frac{1}{2}=\frac{2}{9}$

Equation of AB is $x + y = 1$

$OA = 1,\ AM = \sec(45° - \theta)$

$AN = \sec(45° - \theta)\cos\theta$

$MN = \sec(45° - \theta)\sin\theta$

$$Ar(\Delta AMN) = \frac{1}{2}\times\sec^2(45°-\theta)\sin\theta.\cos\theta=\frac{2}{9}$$

$\Rightarrow \tan\theta = 2,\ \frac{1}{2}$

$\tan\theta = 2$ is rejected

$$\frac{AN}{NB}=\frac{\lambda}{1}=\cot\theta=2$$

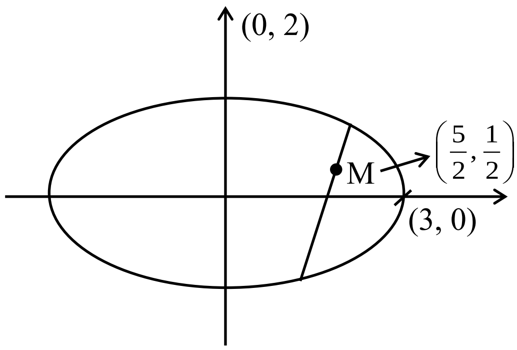
9. If $\alpha x + \beta y = 109$ is the equation of the chord of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$, whose mid point is $\left(\frac{5}{2},\frac{1}{2}\right)$, then $\alpha + \beta$ is equal to

(1) 37 (2) 46

(3) 58 (4) 72

Ans. **(3)**

Sol.



Equation of chord $T = S_1$

$$\frac{5}{2}\left(\frac{x}{9}\right)+\frac{1}{2}\left(\frac{y}{4}\right)=\frac{25}{36}+\frac{1}{16}$$

$$\Rightarrow \frac{5x}{18}+\frac{y}{8}=\frac{100+9}{144}=\frac{109}{144}$$

$\Rightarrow\ 40x + 18y = 109$

$\Rightarrow\ \alpha = 40, \beta = 18$

$\Rightarrow\ \alpha + \beta = 58$

10. If all the words with or without meaning made using all the letters of the word "KANPUR" are arranged as in a dictionary, then the word at 440th position in this arrangement, is :

(1) PRNAKU (2) PRKANU

(3) PRKAUN (4) PRNAUK

Ans. **(3)**

Sol. A, K, N, P, R, U

$\boxed{A}$ .... = $\underline{|5}$ = 120

$\boxed{K}$ .... = $\underline{|5}$ = 120

$\boxed{N}$ .... = $\underline{|5}$ = 120

$\boxed{P}\,\boxed{A}$ .... = $\underline{|4}$ = 24

$\boxed{P}\,\boxed{K}$ .... = $\underline{|4}$ = 24

$\boxed{P}\,\boxed{N}$ .... = $\underline{|4}$ = 24

$\boxed{P}\,\boxed{R}\,\boxed{A}$ .... = $\underline{|3}$ = 6

$\boxed{P}\,\boxed{R}\,\boxed{K}\,\boxed{A}\,\boxed{N}\,\boxed{U}$ = 1

$\boxed{P}\,\boxed{R}\,\boxed{K}\,\boxed{A}\,\boxed{U}\,\boxed{N}$ = 1

Total = 440

$\Rightarrow$ 440$^{\text{th}}$ word

11. Let α,β $(\alpha \neq \beta)$ be the values of m, for which the equations $x + y + z = 1$; $x + 2y + 4z = m$ and $x + 4y + 10z = m^2$ have infinitely many solutions. Then the value of $\sum_{n=1}^{10}\left(n^{\alpha}+n^{\beta}\right)$ is equal to :

(1) 440 (2) 3080

(3) 3410 (4) 560

Ans. **(1)**

Sol. $\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 4 & 10 \end{vmatrix} = 1(20-16) - 1(10-4) + 1(4-2)$

$= 4 - 6 + 2 = 0$

For infinite solutions

$\Delta_x = \Delta_y = \Delta_z = 0$

$m^2 - 3x + 2 = 0$

$m = 1, 2$

$\alpha = 1, \beta = 2$

$\therefore \sum_{n=1}^{10}(n^{\alpha}+n^{\beta}) = \sum_{n=1}^{10} n^1 + \sum_{n=1}^{10} n^2$

$= \frac{10(11)}{2} + \frac{10(11)(21)}{6}$

$= 55 + 385$

$= 440$

12. Let $A = [a_{ij}]$ be a matrix of order 3×3, with $a_{ij} = \left(\sqrt{2}\right)^{i+j}$. If the sum of all the elements in the third row of A^2 is $\alpha + \beta\sqrt{2}$, $\alpha,\beta \in \mathbf{Z}$, then $\alpha+\beta$ is equal to

(1) 280 (2) 168

(3) 210 (4) 224

Ans. **(4)**

Sol. $A = \begin{bmatrix} (\sqrt{2})^2 & (\sqrt{2})^3 & (\sqrt{2})^4 \\ (\sqrt{2})^3 & (\sqrt{2})^4 & (\sqrt{2})^5 \\ (\sqrt{2})^4 & (\sqrt{2})^5 & (\sqrt{2})^6 \end{bmatrix}$

$A = \begin{bmatrix} 2 & 2\sqrt{2} & 4 \\ 2\sqrt{2} & 4 & 4\sqrt{2} \\ 4 & 4\sqrt{2} & 8 \end{bmatrix}$

$A^2 = 2^2 \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 2 & 2\sqrt{2} \\ 2 & 2\sqrt{2} & 4 \end{bmatrix} \begin{bmatrix} 1 & \sqrt{2} & 2 \\ \sqrt{2} & 2 & 2\sqrt{2} \\ 2 & 2\sqrt{2} & 4 \end{bmatrix}$

$= 4\begin{bmatrix} - & - & - \\ - & - & - \\ (2+4+8) & (2\sqrt{2}+4\sqrt{2}+8\sqrt{2}) & (4+8+16) \end{bmatrix}$

Sum of elements of 3$^{\text{rd}}$ row = $4(14 + 14\sqrt{2} + 28)$

$= 4(42 + 14\sqrt{2})$

$= 168 + 56\sqrt{2}$

$\alpha + \beta\sqrt{2}$

$\therefore$ $\alpha + \beta = 168 + 56 = 224$

13. Let P be the foot of the perpendicular from the point (1, 2, 2) on the line L : $\frac{x-1}{1} = \frac{y+1}{-1} = \frac{z-2}{2}$.

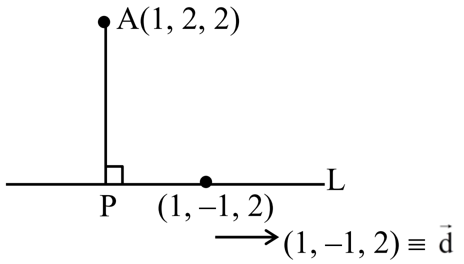
Let the line $\vec{r} = \left(-\hat{i}+\hat{j}-2\hat{k}\right)+\lambda\left(\hat{i}-\hat{j}+\hat{k}\right)$, $\lambda \in \mathbf{R}$, intersect the line L at Q. Then $2(PQ)^2$ is equal to:

(1) 27 (2) 25

(3) 29 (4) 19

Ans. **(1)**

Sol.



$$L : \frac{x-1}{1} = \frac{y+1}{-1} = \frac{z-2}{2} = \mu$$

$P(\mu + 1, -\mu - 1, 2\mu + 2)$

$\overrightarrow{AP} \cdot \vec{d} = 0 \Rightarrow (\mu, -\mu - 3, 2\mu) \cdot (1, -1, 2) = 0$

$\Rightarrow \mu + \mu + 3 + 4\mu = 0 \Rightarrow \mu = -\frac{1}{2}$

$\therefore\ P\left(\frac{-1}{2} + 1, +\frac{1}{2} - 1, 2\left(\frac{-1}{2}\right) + 2\right)$

$P\left(\frac{1}{2}, \frac{-1}{2}, 1\right)$

Now general pt. on L_2 is $Q(-1 + \lambda, 1 - \lambda, -2 + \lambda)$

Equate it with general pt of L

$\mu + 1 = -1 + \lambda$	$-\mu - 1 = 1 - \lambda$	$2\mu + 2 = -2 + \lambda$
$\mu = \lambda - 2$	$\mu = \lambda - 2$	$\downarrow$

$2(\lambda - 2) + 2 = -2 + \lambda$

$2\lambda - 4 + 2 = -2 + \lambda$

$\therefore\ \mu = -2\ ,\ \lambda = 0$

$\therefore\ Q \equiv (-1, 1 -2)$

$P\left(\frac{1}{2}, \frac{-1}{2}, 1\right)$ and $Q(-1, 1, -2)$

$$PQ = \sqrt{\left(\frac{1}{2}+1\right)^2 + \left(\frac{-1}{2}-1\right)^2 + (1+2)^2}$$

$$= \sqrt{\frac{9}{4} + \frac{9}{4} + 9} = \sqrt{\frac{54}{4}}$$

$\therefore\ 2(PQ)^2 = 2\left(\frac{54}{4}\right) = 27$

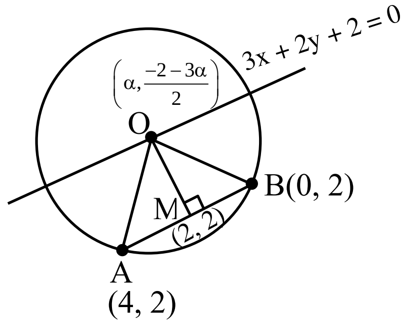
14. Let a circle C pass through the points (4, 2) and (0, 2), and its centre lie on $3x + 2y + 2 = 0$. Then the length of the chord, of the circle C, whose mid-point is (1, 2), is:

(1) $\sqrt{3}$ (2) $2\sqrt{3}$

(3) $4\sqrt{2}$ (4) $2\sqrt{2}$

Ans. (2)

Sol.



$M_{AB} = 0 \Rightarrow$ OM is vertical

$\Rightarrow\ \alpha = 2$

$\therefore$ Centre (0) ≡ (2, –4)

$r = OA = \sqrt{(2-4)^2 + (2+4)^2} = \sqrt{40}$

mid point of chord is N ≡ (1, 2) $\therefore$ ON = $\sqrt{37}$

$\therefore$ length of chord = $2\sqrt{r^2 - (ON)^2}$

$= 2\sqrt{40 - 37} = 2\sqrt{3}$

15. Let $A = [a_{ij}]$ be a 2×2 matrix such that $a_{ij} \in \{0, 1\}$ for all i and j. Let the random variable X denote the possible values of the determinant of the matrix A. Then, the variance of X is:

(1) $\frac{1}{4}$ (2) $\frac{3}{8}$

(3) $\frac{5}{8}$ (4) $\frac{3}{4}$

Ans. (2)

Sol. $|A| = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix}$

$= a_{11}a_{22} - a_{21}a_{12}$

$= \{-1, 0, 1\}$

x	P_i	P_iX_i	$P_1X_i^2$
–1	$\frac{3}{16}$	$-\frac{3}{16}$	$\frac{3}{16}$
0	$\frac{10}{16}$	0	0
1	$\frac{3}{16}$	$\frac{3}{16}$	$\frac{3}{16}$
		$\Sigma P_iX_i = 0$	$\Sigma P_iX_i^2 = \frac{3}{8}$

$\therefore$ var(x) = $\Sigma P_iX_i^2 - (\Sigma P_iX_i)^2$

$= \frac{3}{8} - 0 = \frac{3}{8}$

16. Bag 1 contains 4 white balls and 5 black balls, and Bag 2 contains n white balls and 3 black balls. One ball is drawn randomly from Bag 1 and transferred to Bag 2. A ball is then drawn randomly from Bag 2. If the probability, that the ball drawn is white, is 29/45, then n is equal to:

(1) 3 (2) 4

(3) 5 (4) 6

Ans. **(4)**

Sol. Bag 1 = {4W, 5B}

Bag 2 = {nW, 3B}

$$P\left(\frac{W}{\text{Bag}\,2}\right)=\frac{29}{45}$$

$$\Rightarrow P\left(\frac{W}{B_1}\right)\times P\left(\frac{W}{B_2}\right)+P\left(\frac{B}{B_1}\right)\times P\left(\frac{W}{B_2}\right)=\frac{29}{45}$$

$$\frac{4}{9}\times\frac{n+1}{n+4}+\frac{5}{9}\times\frac{n}{n+4}=\frac{29}{45}$$

$$\boxed{n=6}$$

17. The remainder, when 7^{103} is divided by 23, is equal to:

(1) 14 (2) 9

(3) 17 (4) 6

Ans. **(1)**

Sol. $7^{103} = 7\,(7^{102}) = 7\,(343)^{34} = 7\,(345-2)^{34}$

$7^{103} = 23K_1 + 7.2^{34}$

Now $7.2^{34} = 7\,.\,2^2\,.\,2^{32}$

$= 28\,.\,(256)^4$

$= 28\,(253+3)^4$

$\therefore\ 28 \times 81 \Rightarrow (23+5)\,(69+12)$

$23K_2 + 60$

$\therefore$ Remainder = $\boxed{14}$

18. Let $f(x)=\int_0^x t\left(t^2-9t+20\right)dt,\ \ 1 \le x \le 5$. If the range of f is $[\alpha, \beta]$, then $4(\alpha+\beta)$ equals:

(1) 157 (2) 253

(3) 125 (4) 154

Ans. **(1)**

Sol. $f'(x) = x^3 - 9x^2 + 20x = x(x-4)\,(x-5)$

$$\begin{array}{ccccccc} - & & + & & - & & + \\ \hline & 0 & & 4 & & 5 & \end{array}$$

$$\therefore\ \mathbf{f(x)} = \frac{x^4}{4}-\frac{9x^3}{3}+\frac{20x^2}{2}$$

$$f(1) = \frac{1}{4} - 3 + 10 = \frac{29}{4} = \alpha$$

$$f(4) = \frac{256}{4} - 3(64) + 10(16) = 32 = \beta$$

$$4(\alpha+\beta) = 4\left(\frac{29}{4}+32\right) = 157$$

19. Let $\hat{a}$ be a unit vector perpendicular to the vectors $\vec{b}=\hat{i}-2\hat{j}+3\hat{k}$ and $\vec{c}=2\hat{i}+3\hat{j}-\hat{k}$, and makes an angle of $\cos^{-1}\left(-\frac{1}{3}\right)$ with the vector $\hat{i}+\hat{j}+\hat{k}$. If $\hat{a}$ makes an angle of $\frac{\pi}{3}$ with the vector $\hat{i}+\alpha\hat{j}+\hat{k}$, then the value of α is :

(1) $-\sqrt{3}$ (2) $\sqrt{6}$

(3) $-\sqrt{6}$ (4) $\sqrt{3}$

Ans. **(3)**

Sol. Let $\vec{v}=\hat{i}+\hat{j}+\hat{k}$

$$\vec{b}\times\vec{c}=\begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 1 & -2 & 3\\ 2 & 3 & -1\end{vmatrix}$$

$$=-7\hat{i}+7\hat{j}+7\hat{k}$$

$$=-7(\hat{i}-\hat{j}-\hat{k})$$

Now $\hat{a}=\dfrac{\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}$ or $\dfrac{-\hat{i}+\hat{j}+\hat{k}}{\sqrt{3}}$

$\downarrow$ $\downarrow$

$$\cos\theta=\frac{\hat{a}.\vec{v}}{|\vec{v}|}=\frac{1-1-1}{\sqrt{3}\sqrt{3}}=\frac{-1}{3} \qquad \cos\theta=\frac{\hat{a}.\vec{v}}{|\vec{v}|}=\frac{-1+1+1}{3}=\frac{1}{3}$$

(rejected)

$$\Rightarrow \hat{a}=\frac{\hat{i}-\hat{j}-\hat{k}}{\sqrt{3}}$$

Now $\cos\frac{\pi}{3}=\frac{\hat{a}\cdot(\hat{i}+\alpha\hat{j}+\hat{k})}{\sqrt{1+\alpha^2+1}}$

$\Rightarrow\frac{1}{2}=\frac{1-\alpha-1}{\sqrt{3}\sqrt{\alpha^2+2}}$

$\Rightarrow\frac{\sqrt{3}}{2}\sqrt{\alpha^2+2}=-\alpha \quad (\because\ \alpha<0)$

$3\alpha^2 + 6 = 4\alpha^2$

$\Rightarrow \alpha = -\sqrt{6}$

20. If for the solution curve y = f(x) of the differential equation $\frac{dy}{dx}+(\tan x)y=\frac{2+\sec x}{(1+2\sec x)^2}$,

$x\in\left(\frac{-\pi}{2},\frac{\pi}{2}\right), f\left(\frac{\pi}{3}\right)=\frac{\sqrt{3}}{10}$, then $f\left(\frac{\pi}{4}\right)$ is equal to:

(1) $\frac{9\sqrt{3}+3}{10(4+\sqrt{3})}$ (2) $\frac{\sqrt{3}+1}{10(4+\sqrt{3})}$

(3) $\frac{5-\sqrt{3}}{2\sqrt{2}}$ (4) $\frac{4-\sqrt{2}}{14}$

Ans. (4)

Sol. If $e^{\int\tan x\,dx}=e^{\ell n(\sec x)}=\sec x$

$\therefore$ **y . sec x** $=\int\left\{\frac{2+\sec x}{(1+2\sec x)^2}\right\}\sec x\,dx$

$=\int\frac{2\cos x+1}{(\cos x+2)^2}dx$ Let $\cos x=\frac{1-t^2}{1+t^2}$

$=\int\frac{2\left(\frac{1-t^2}{1+t^2}\right)+1}{\left(\frac{1-t^2}{1+t^2}+2\right)^2}2dt$

$=\int\frac{2-2t^2+1+t^2}{(1-t^2+2+2t^2)^2}\times 2dt$

$=2\int\frac{3-t^2}{(t^2+3)^2}dt$

Let $t+\frac{3}{t}=u$

$\left(1-\frac{3}{t^2}\right)dt=du$

$=-2\int\frac{du}{u^2}$

y . (sec x) $=\frac{2}{u}+c$

$$\boxed{y.\sec x=\frac{2}{t+\frac{3}{t}}+c} \quad ....(I)$$

At $x=\frac{\pi}{3}$, $t=\tan\frac{x}{2}=\frac{1}{\sqrt{3}}$

$2.\frac{\sqrt{3}}{10}=\frac{2}{\frac{1}{\sqrt{3}}+3\sqrt{3}}+c$

$2.\frac{\sqrt{3}}{10}=\frac{2\sqrt{3}}{10}+c\Rightarrow C=0$

At $x=\frac{\pi}{4}$, $t=\tan\frac{x}{2}=\sqrt{2}-1$

$\therefore\ y.\sqrt{2}=\frac{2}{\sqrt{2}-1+\frac{3}{\sqrt{2}-1}}$

$y.\sqrt{2}=\frac{2(\sqrt{2}-1)}{6-2\sqrt{2}}$

$y=\frac{\sqrt{2}(\sqrt{2}-1)}{2(3-\sqrt{2})}=\frac{1}{\sqrt{2}}\times\frac{2\sqrt{2}-1}{7}$

$=\frac{4-\sqrt{2}}{14}$

SECTION-B

21. If $24\int_0^{\frac{\pi}{4}}\left(\sin\left|4x-\frac{\pi}{12}\right|+[2\sin x]\right)dx = 2\pi+\alpha$, where [·] denotes the greatest integer function, then α is equal to ________.

Ans. (12)

Sol. $=24\int_0^{\frac{\pi}{48}}-\sin\left(4x-\frac{\pi}{12}\right)+\int_{\pi/48}^{\pi/4}\sin\left(4x-\frac{\pi}{12}\right)$

$+\int_0^{\frac{\pi}{6}}[0]dx+\int_{\pi/6}^{\pi/4}[2\underset{\downarrow\atop 1}{\sin x}]dx$

$=24\left[\frac{\left(1-\cos\frac{\pi}{12}\right)}{4}-\frac{\left(-\cos\frac{\pi}{12}-1\right)}{4}\right]+\frac{\pi}{4}-\frac{\pi}{6}$

$=24\left(\frac{1}{2}+\frac{\pi}{12}\right)=2\pi+12$

$\alpha=12$

22. If $\lim\limits_{t\to 0}\left(\int_0^1 (3x+5)^t dx\right)^{\frac{1}{t}} = \frac{\alpha}{5e}\left(\frac{8}{5}\right)^{\frac{2}{3}}$, then α is equal to ________.

Ans. **(64)**

Sol. 1^∞ form

$$\text{Now } L = e^{\lim\limits_{t\to 0}\frac{1}{t}\left(\left.\frac{(3x+5)^{t+1}}{3(t+1)}\right|_0^1 - 1\right)}$$

$$= e^{\lim\limits_{t\to 0}\frac{8^{t+1}-5^{t+1}-3t-3}{3t(t+1)}}$$

$$= e^{\frac{8\ell n8 - 5\ell n5 - 3}{3}}$$

$$= \left(\frac{8}{5}\right)^{2/3}\left(\frac{64}{5}\right) = \frac{\alpha}{5e}\left(\frac{8}{5}\right)^{2/3}$$

On comparing

$\alpha = 64$

23. Let $a_1, a_2, \ldots, a_{2024}$ be an Arithmetic Progression such that $a_1 + (a_5 + a_{10} + a_{15} + \ldots + a_{2020}) + a_{2024} = 2233$. Then $a_1 + a_2 + a_3 + \ldots + a_{2024}$ is equal to ________.

Ans. **(11132)**

Sol. $a_1 + a_5 + a_{10} + \ldots + a_{2020} + a_{2024} = 2233$

In an A.P. the sum of terms equidistant from ends is equal.

$a_1 + a_{2024} = a_5 + a_{2020} = a_{10} + a_{2015} \ldots$

$\Rightarrow$ 203 pairs

$\Rightarrow 203(a_1 + a_{2024}) = 2233$

Hence,

$$S_{2024} = \frac{2024}{2}(a_1 + a_{2024})$$

$= 1012 \times 11$

$= 11132$

24. Let integers $a, b \in [-3, 3]$ be such that $a + b \neq 0$. Then the number of all possible ordered pairs (a, b), for which $\left|\frac{z-a}{z+b}\right| = 1$ and $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$, $z \in C$, where ω and ω^2 are the roots of $x^2 + x + 1 = 0$, is equal to ________.

Ans. **(10)**

Sol. $a, b \in I, -3 \leq a, b \leq 3, a + b \neq 0$

$|z - a| = |z + b|$

$$\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$$

$$\Rightarrow \begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$$

$$\Rightarrow z\begin{vmatrix} 1 & 1 & 1 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 1$$

$$\Rightarrow z\begin{vmatrix} 1 & 0 & 0 \\ \omega & z+\omega^2-\omega & 1-\omega \\ \omega^2 & 1-\omega^2 & z+\omega-\omega^2 \end{vmatrix} = 1$$

$\Rightarrow z^3 = 1$

$\Rightarrow z = \omega, \omega^2, 1$

Now

$|1-a| = |1+b|$

$\Rightarrow$ 10 pairs

25. Let $y^2 = 12x$ the parabola and S be its focus. Let PQ be a focal chord of the parabola such that (SP) (SQ) $= \frac{147}{4}$. Let C be the circle described taking PQ as a diameter. If the equation of a circle C is $64x^2 + 64y^2 - \alpha x - 64\sqrt{3}y = \beta$, then $\beta - \alpha$ is equal to __________.

Ans. **(1328)**

Sol. $y^2 = 12x \quad a = 3 \quad SP \times SQ = \frac{147}{4}$

Let $P(3t^2, 6t)$ and $t_1t_2 = -1$

(ends of focal chord)

So, $Q\left(\frac{3}{t^2}, \frac{-6}{t}\right)$

$S(3, 0)$

$SP \times SQ = PM_1 \times QM_2$

(dist. from directrix)

$= (3 + 3t^2)\left(3+\frac{3}{t^2}\right) = \frac{147}{4}$

$\Rightarrow \frac{(1+t^2)^2}{t^2} = \frac{49}{12}$

$t^2 = \frac{3}{4}, \frac{4}{3}$

$t = \pm\frac{\sqrt{3}}{2}, \pm\frac{2}{\sqrt{3}}$

considering $t = \frac{-\sqrt{3}}{2}$

$P\left(\frac{9}{4}, -3\sqrt{3}\right)$ and $Q\left(4, 4\sqrt{3}\right)$

Hence, diametric circle:

$(x-4)\left(x-\frac{9}{4}\right) + (y+3\sqrt{3})\,(y-4\sqrt{3}) = 0$

$\Rightarrow x^2 + y^2 - \frac{25}{4}x - \sqrt{3}y - 27 = 0$

$\Rightarrow \alpha = 400, \beta = 1728$

$\beta - \alpha = 1328$