

JEE–MAIN EXAMINATION – JANUARY 2025

(HELD ON WEDNESDAY 29TH JANUARY 2025)

TIME : 9:00 AM TO 12:00 NOON

MATHEMATICS

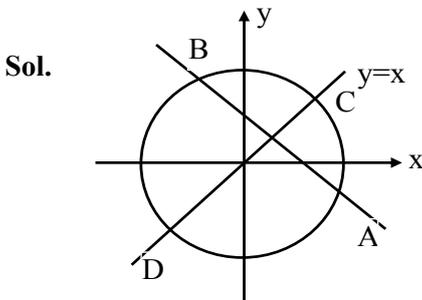
TEST PAPER WITH SOLUTION

SECTION-A

1. Let the line $x + y = 1$ meet the circle $x^2 + y^2 = 4$ at the points A and B. If the line perpendicular to AB and passing through the mid point of the chord AB intersects the circle at C and D, then the area of the quadrilateral ADCB is equal to

- (1) $3\sqrt{7}$ (2) $2\sqrt{14}$
 (3) $5\sqrt{7}$ (4) $\sqrt{14}$

Ans. (2)



Sol.

By solving $x = y$ with circle

We get

$$C(\sqrt{2}, \sqrt{2})$$

$$D(-\sqrt{2}, -\sqrt{2})$$

By solving $x + y = 1$ with circle $x^2 + y^2 = 4$

we set

$$A\left(\frac{1+\sqrt{7}}{2}, \frac{1-\sqrt{7}}{2}\right)$$

$$\& B\left(\frac{1-\sqrt{7}}{2}, \frac{1+\sqrt{7}}{2}\right)$$

\therefore Area of Quadrilateral ACBD

= 2 \times Area of ΔBCD

$$= 2 \times \frac{1}{2} \begin{vmatrix} \sqrt{2} & \sqrt{2} & 1 \\ \frac{1-\sqrt{7}}{2} & \frac{1+\sqrt{7}}{2} & 1 \\ -\sqrt{2} & -\sqrt{2} & 1 \end{vmatrix}$$

$$= 2\sqrt{14}$$

2. Let M and m respectively be the maximum and the minimum values of

$$f(x) = \begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4 \sin 4x \\ \sin^2 x & 1 + \cos^2 x & 4 \sin 4x \\ \sin^2 x & \cos^2 x & 1 + 4 \sin 4x \end{vmatrix}, x \in \mathbb{R}$$

Then $M^4 - m^4$ is equal to :

- (1) 1280 (2) 1295
 (3) 1040 (4) 1215

Ans. (1)

Sol.
$$\begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4 \sin 4x \\ \sin^2 x & 1 + \cos^2 x & 4 \sin 4x \\ \sin^2 x & \cos^2 x & 1 + 4 \sin 4x \end{vmatrix}, x \in \mathbb{R}$$

$$R_2 \rightarrow R_2 - R_1 \ \& \ R_3 \rightarrow R_3 - R_1$$

$$f(x) \begin{vmatrix} 1 + \sin^2 x & \cos^2 x & 4 \sin 4x \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{vmatrix}$$

Expand about R_1 , use get

$$f(x) = 2 + 4 \sin 4x$$

$$\therefore M = \text{max value of } f(x) = 6$$

$$M = \text{min value of } f(x) = -2$$

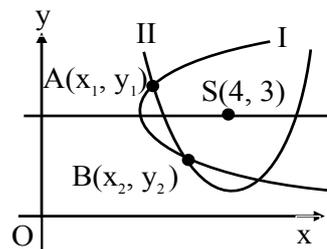
$$\therefore M^4 - m^4 = 1280$$

3. Two parabolas have the same focus (4,3) and their directrices are the x-axis and the y-axis, respectively. If these parabolas intersect at the points A and B, then $(AB)^2$ is equal to

- (1) 192 (2) 384
 (3) 96 (4) 392

Ans. (1)

Sol.



Sol. Consider $\frac{1}{\sqrt{x}} = \alpha$ $x > 0$

$$\{9\alpha^2 - 9\alpha + 2\} \{2\alpha^2 - 7\alpha + 3\} = 0$$

$$(3\alpha - 2)(3\alpha - 1)(\alpha - 3)(2\alpha - 1) = 0$$

$$\alpha = \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, 3$$

$$x = 9, 4, \frac{9}{4}, \frac{1}{9}$$

So, no. of solutions = 4

10. Let $y = y(x)$ be the solution of the differential equation

$$\cos x (\log_e(\cos x))^2 dy + (\sin x - 3y \sin x \log_e(\cos x)) dx = 0,$$

$x \in \left(0, \frac{\pi}{2}\right)$. If $y\left(\frac{\pi}{4}\right) = \frac{-1}{\log_e 2}$, then $y\left(\frac{\pi}{6}\right)$ is :

(1) $\frac{2}{\log_e(3) - \log_e(4)}$ (2) $\frac{1}{\log_e(4) - \log_e(3)}$

(3) $-\frac{1}{\log_e(4)}$ (4) $\frac{1}{\log_e(3) - \log_e(4)}$

Ans. (4)

Sol.

$$\cos x (\ln(\cos x))^2 dy + (\sin x - 3y \sin x \ln(\cos x)) dx = 0$$

$$\cos x (\ln(\cos x))^2 \frac{dy}{dx} - 3 \sin x \ln(\cos x) y = -\sin x$$

$$\frac{dy}{dx} - \frac{3 \tan x}{\ln(\cos x)} y = \frac{-\tan x}{(\ln(\cos x))^2}$$

$$\frac{dy}{dx} + \frac{3 \tan x}{\ln(\sec x)} y = \frac{-\tan x}{(\ln(\sec x))^2}$$

$$\text{I.F.} = e^{\int \frac{3 \tan x}{\ln(\sec x)} dx} = (\ln(\sec x))^3$$

$$y \times (\ln(\sec x))^3 = -\int \frac{\tan x}{(\ln(\sec x))^2} (\ln(\sec x))^3 dx + C$$

$$y \times (\ln(\sec x))^3 = -\frac{1}{2} (\ln(\sec x))^2 + C$$

Given : $x = \frac{\pi}{4}$, $y = -\frac{1}{\ln 2}$

$$\frac{-1}{\ln 2} \times (\ln \sqrt{2})^3 = -\frac{1}{2} \times (\ln \sqrt{2})^2 + C$$

$$\Rightarrow \frac{-1}{8 \ln 2} \times (\ln 2)^3 = \frac{-1}{2} \times \frac{1}{4} (\ln 2)^2 + C$$

$$-\frac{1}{8} (\ln 2)^2 = \frac{-1}{8} (\ln 2)^2 + C$$

$$\Rightarrow C = 0$$

$$\therefore y (\ln(\sec x))^3 = \frac{-1}{2} (\ln(\sec x))^2 + 0$$

$$y = \frac{-1}{2 \ln(\sec x)}$$

$$y = \frac{1}{2 \ln(\cos x)}$$

$$\therefore y\left(\frac{\pi}{6}\right) = \frac{1}{2 \ln\left(\cos \frac{\pi}{6}\right)}$$

$$= \frac{1}{2 \ln\left(\frac{\sqrt{3}}{2}\right)}$$

$$= \frac{1}{2\left(\frac{1}{2} \ln 3 - \ln 2\right)}$$

$$= \frac{1}{\ln 3 - \ln 4}$$

Option (4)

11. Define a relation R on the interval $\left[0, \frac{\pi}{2}\right)$ by $x R y$

if and only if $\sec^2 x - \tan^2 y = 1$. Then R is :

- (1) an equivalence relation
- (2) both reflexive and transitive but not symmetric
- (3) both reflexive and symmetric but not transitive
- (4) reflexive but neither symmetric nor transitive

Ans. (1)

Sol. $\sec^2 x - \tan^2 x = 1$ (on replacing y with x)

\Rightarrow Reflexive

$$\sec^2 x - \tan^2 y = 1$$

$$\Rightarrow 1 + \tan^2 x + 1 - \sec^2 y = 1$$

$$\Rightarrow \sec^2 y - \tan^2 x = 1$$

\Rightarrow symmetric

$$\sec^2 x - \tan^2 y = 1,$$

$$\sec^2 y - \tan^2 z = 1$$

Adding both

$$\Rightarrow \sec^2 x - \tan^2 y + \sec^2 y - \tan^2 z = 1 + 1$$

$$\sec^2 x + 1 - \tan^2 z = 2$$

$$\sec^2 x - \tan^2 z = 1$$

$$\Rightarrow \text{Transitive}$$

hence equivalence relation

Option (1)

12. Let the ellipse, $E_1 : \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, $a > b$ and

$$E_2 : \frac{x^2}{A^2} + \frac{y^2}{B^2} = 1, A < B \text{ have same eccentricity}$$

$\frac{1}{\sqrt{3}}$. Let the product of their lengths of latus

rectums be $\frac{32}{\sqrt{3}}$, and the distance between the foci

of E_1 be 4. If E_1 and E_2 meet at A,B,C and D, then the area of the quadrilateral ABCD equals:

- (1) $6\sqrt{6}$ (2) $\frac{18\sqrt{6}}{5}$
 (3) $\frac{12\sqrt{6}}{5}$ (4) $\frac{24\sqrt{6}}{5}$

Ans. (4)

Sol. $2ae = 4$

$$2a\left(\frac{1}{\sqrt{3}}\right) = 4$$

$$\Rightarrow a = 2\sqrt{3}$$

$$\Rightarrow 1 - \frac{b^2}{12} = \frac{1}{3} \Rightarrow b^2 = 8$$

$$\text{Now } \frac{2b^2}{a} \cdot \frac{2A^2}{B} = \frac{32}{\sqrt{3}} \Rightarrow 2\left(\frac{8}{2\sqrt{3}}\right) \frac{2A^2}{B} = \frac{32}{\sqrt{3}}$$

$$\Rightarrow A^2 = 2B$$

$$1 - \frac{A^2}{B^2} = \frac{1}{3} \Rightarrow 1 - \frac{2B}{B^2} = \frac{1}{3} \Rightarrow B = 3$$

$$\Rightarrow A^2 = 6$$

$$\frac{x^2}{12} + \frac{y^2}{8} = 1 \dots\dots(1)$$

$$\frac{x^2}{6} + \frac{y^2}{9} = 1 \dots\dots(2)$$

On solving (1) & (2) we get

$$(x, y) \equiv \left(\frac{\sqrt{6}}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right), \left(\frac{-\sqrt{6}}{\sqrt{5}}, \frac{6}{\sqrt{5}}\right), \left(\frac{\sqrt{6}}{\sqrt{5}}, \frac{-6}{\sqrt{5}}\right), \left(\frac{-\sqrt{6}}{\sqrt{5}}, \frac{-6}{\sqrt{5}}\right)$$

The four points are vertices of rectangle and its area = $\frac{24\sqrt{6}}{5}$

13. Consider an A.P. of positive integers, whose sum of the first three terms is 54 and the sum of the first twenty terms lies between 1600 and 1800. Then its 11th term is :

- (1) 84 (2) 122
 (3) 90 (4) 108

Ans. (3)

Sol. $S_3 = 3a + 3d = 54$

$$\Rightarrow a + d = 18$$

$$S_{20} = 10(2a + 19d)$$

$$\Rightarrow 10(36 + 17d)$$

$$\Rightarrow 1600 < 10(36 + 17d) < 1800$$

$$\Rightarrow 160 < 36 + 17d < 180$$

$$\Rightarrow 124 < 17d < 144$$

$$\Rightarrow 7\frac{5}{17} < d < 8\frac{8}{17}$$

Common difference will be natural number

$$\Rightarrow d = 8 \Rightarrow a = 10$$

$$\Rightarrow a_{11} = 10 + 10 \times 8 = 90$$

14. Let $\vec{a} = \hat{i} + 2\hat{j} + \hat{k}$ and $\vec{b} = 2\hat{i} + 7\hat{j} + 3\hat{k}$. Let

$$L_1 : \vec{r} = (-\hat{i} + 2\hat{j} + \hat{k}) + \lambda\vec{a}, \lambda \in \mathbb{R} \text{ and}$$

$$L_2 : \vec{r} = (\hat{j} + \hat{k}) + \mu\vec{b}, \mu \in \mathbb{R} \text{ be two lines. If the}$$

line L_3 passes through the point of intersection of L_1 and L_2 , and is parallel to $\vec{a} + \vec{b}$, then L_3 passes through the point:

- (1) (8, 26, 12) (2) (2, 8, 5)
 (3) (-1, -1, 1) (4) (5, 17, 4)

Ans. (1)

Sol. $L_1 : \vec{r} = (-\hat{i} + 2\hat{j} + \hat{k}) + \lambda(\hat{i} + 2\hat{j} + \hat{k})$

$$\Rightarrow \vec{r} = (\lambda - 1)\hat{i} + 2(\lambda + 1)\hat{j} + (\lambda + 1)\hat{k}$$

$$L_2 : \vec{r} = (\hat{j} + \hat{k}) + \mu(2\hat{i} + 7\hat{j} + 3\hat{k})$$

$$\Rightarrow \vec{r} = 2\mu\hat{i} + (1 + 7\mu)\hat{j} + (1 + 3\mu)\hat{k}$$

$$B(k+1, -k+2, 2k+1)$$

Now

$$\alpha - 5\lambda = k+1 \Rightarrow \alpha = 5\lambda + k+1$$

$$\beta - 3\lambda = -k+2 \Rightarrow \beta = 3\lambda - k+2$$

$$\gamma + \lambda = 2k-1 \Rightarrow \gamma = -\lambda + 2k+1$$

$$|5\alpha - 11\beta - 8\gamma| = |-25| \\ = 25$$

18. Let x_1, x_2, \dots, x_{10} be ten observations such that

$$\sum_{i=1}^{10} (x_i - 2) = 30, \quad \sum_{i=1}^{10} (x_i - \beta)^2 = 98, \quad \beta > 2 \text{ and}$$

their variance is $\frac{4}{5}$. If μ and σ^2 are respectively the

mean and the variance of $2(x_1-1) + 4\beta, 2(x_2-1) +$

$4\beta, \dots, 2(x_{10}-1) + 4\beta$, then $\frac{\beta\mu}{\sigma^2}$ is equal to :

(1) 100

(2) 110

(3) 120

(4) 90

Ans. (1)

Sol. $\frac{4}{5} = \frac{\sum x_i^2}{10} - \left(\frac{\sum x_i}{10}\right)^2$

$$\frac{4}{5} = \frac{\sum x_i^2}{10} - 25$$

$$\Rightarrow \sum x_i^2 = 258$$

Now $\sum_{i=1}^{10} (x_i - \beta)^2 = 98$

$$\sum_{i=1}^{10} (x_i^2 - 2\beta x_i + \beta^2) = 98$$

$$258 - 2\beta(50) + 10\beta^2 = 98$$

$$(\beta - 8)(\beta - 2) = 0$$

$$\beta = 8 \text{ or } \beta = 2 \quad (\text{as } \beta > 2)$$

$$\therefore \beta = 8$$

Now,

$$= 2(x_1-1) + 4\beta, 2(x_2-1) + 4\beta, \dots, 2(x_{10}-1) + 4\beta$$

$$= 2x_1 + 30, 2x_2 + 30, \dots, 2x_{10} + 30$$

$$\mu = 2(5) + 30 = 40$$

$$\sigma^2 = 2^2 \left(\frac{4}{5}\right) = \frac{16}{5}$$

$$\therefore \frac{\beta\mu}{\sigma^2} = \frac{8 \times 40}{16/5} = 100$$

19. Let $|z_1 - 8 - 2i| \leq 1$ and $|z_2 - 2 + 6i| \leq 2$,

$z_1, z_2 \in \mathbb{C}$. Then the minimum value of $|z_1 - z_2|$

is :

(1) 3

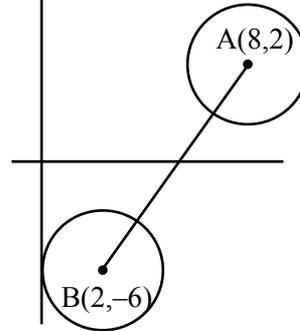
(2) 7

(3) 13

(4) 10

Ans. (2)

Sol.



$$\therefore AB = \sqrt{100} = 10$$

$$\therefore |Z_1 - Z_2|_{\min} = 10 - 2 - 1 = 7$$

20. Let $A = [a_{ij}] = \begin{bmatrix} \log_5 128 & \log_4 5 \\ \log_5 8 & \log_4 25 \end{bmatrix}$.

If A_{ij} is the cofactor of a_{ij} , $C_{ij} = \sum_{k=1}^2 a_{ik} A_{jk}$, $1 \leq i,$

$j \leq 2$, and $C = [C_{ij}]$, then $8|C|$ is equal to :

(1) 262

(2) 288

(3) 242

(4) 222

Ans. (3)

$$|A| = \frac{11}{2}$$

$$C_{11} = \sum_{k=1}^2 a_{1k} \cdot A_{1k} = a_{11}A_{11} + a_{12}A_{12} = |A| = \frac{11}{2}$$

$$C_{12} = \sum_{k=1}^2 a_{1k} \cdot A_{2k} = 0$$

$$C_{21} = \sum_{k=1}^2 a_{2k} \cdot A_{1k} = 0$$

$$C_{22} = \sum_{k=1}^2 a_{2k} \cdot A_{2k} = |A| = \frac{11}{2}$$

$$C = \begin{bmatrix} 11/2 & 0 \\ 0 & 11/2 \end{bmatrix}$$

$$|C| = \frac{121}{4}$$

$$8|C| = 242$$

SECTION-B

21. Let $f : (0, \infty) \rightarrow \mathbb{R}$ be a twice differentiable

function. If for some $a \neq 0$, $\int_0^1 f(\lambda x) d\lambda = af(x)$,

$f(1) = 1$ and $f(16) = \frac{1}{8}$, then $16 - f'\left(\frac{1}{16}\right)$ is equal

to _____,

Ans. (112)

Sol. $\int_0^1 f(\lambda x) d\lambda = af(x)$

$$\lambda x = t$$

$$d\lambda = \frac{1}{x} dt$$

$$\frac{1}{x} \int_0^x f(t) dt = af(x)$$

$$\int_0^x f(t) dt = axf(x)$$

$$f(x) = a(x f'(x) + f(x))$$

$$(1 - a)f(x) = a.x f'(x)$$

$$\frac{f'(x)}{f(x)} = \frac{(1 - a)}{a} \frac{1}{x}$$

$$\ln f(x) = \frac{1 - a}{a} \ln x + c$$

$$x = 1, f(1) = 1 \Rightarrow c = 0$$

$$x = 16, f(16) = \frac{1}{8}$$

$$\frac{1}{8} = (16)^{\frac{1-a}{a}} \Rightarrow -3 = \frac{4-4a}{a} \Rightarrow a = 4$$

$$f(x) = x^{-\frac{3}{4}}$$

$$f'(x) = -\frac{3}{4} x^{-\frac{7}{4}}$$

$$\therefore 16 - f'\left(\frac{1}{16}\right)$$

$$= 16 - \left(-\frac{3}{4}(2^{-4})^{-7/4}\right)$$

$$= 16 + 96 = 112$$

22. Let $S = \left\{m \in \mathbb{Z} : A^{m^2} + A^m = 3I - A^{-6}\right\}$, where

$$A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}. \text{ Then } n(S) \text{ is equal to } \underline{\hspace{2cm}}.$$

Ans. (2)

Sol. $A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 3 & -2 \\ 2 & -1 \end{bmatrix}, A^3 = \begin{bmatrix} 4 & -3 \\ 3 & -2 \end{bmatrix}, A^4 = \begin{bmatrix} 5 & -4 \\ 4 & -3 \end{bmatrix}$$

and so on

$$A^6 = \begin{bmatrix} 7 & -6 \\ 6 & -5 \end{bmatrix}$$

$$A^m = \begin{bmatrix} m+1 & -m \\ m & -m-1 \end{bmatrix},$$

$$A^{m^2} = \begin{bmatrix} m^2+1 & -m^2 \\ m^2 & -(m^2-1) \end{bmatrix}$$

$$A^{m^2} + A^m = 3I - A^{-6}$$

$$\begin{bmatrix} m+1 & -m^2 \\ m^2 & -(m^2-1) \end{bmatrix} + \begin{bmatrix} m+1 & -m \\ m & -(m-1) \end{bmatrix}$$

$$= 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} -5 & 6 \\ -6 & 7 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & -6 \\ 6 & -4 \end{bmatrix}$$

$$= m^2 + 1 + m + 1 = 8$$

$$= m^2 + m - 6 = 0 \Rightarrow m = -3, 2$$

$$n(S) = 2$$

23. Let $[t]$ be the greatest integer less than or equal to t .

Then the least value of $p \in \mathbb{N}$ for which

$$\lim_{x \rightarrow 0^+} \left(x \left(\left[\frac{1}{x} \right] + \left[\frac{2}{x} \right] + \dots + \left[\frac{p}{x} \right] \right) - x^2 \left(\left[\frac{1}{x^2} \right] + \left[\frac{2^2}{x^2} \right] + \dots + \left[\frac{9^2}{x^2} \right] \right) \right) \geq 1$$

is equal to _____.

Ans. (24)

Sol. $\lim_{x \rightarrow 0^+} \left(x \left(\left[\frac{1}{x} \right] + \left[\frac{2}{x} \right] + \dots + \left[\frac{p}{x} \right] \right) - x^2 \left(\left[\frac{1}{x^2} \right] + \left[\frac{2^2}{x^2} \right] + \left[\frac{9^2}{x^2} \right] \right) \right) \geq 1$

$$(1 + 2 + \dots + p) - (1^2 + 2^2 + \dots + 9^2) \geq 1$$

$$\frac{p(p+1)}{2} - \frac{9 \cdot 10 \cdot 19}{6} \geq 1$$

$$p(p+1) \geq 572$$

Least natural value of p is 24

- 24.** The number of 6-letter words, with or without meaning, that can be formed using the letters of the word MATHS such that any letter that appears in the word must appear at least twice, is 4 ____.

Ans. (1405)

- Sol.** (i) Single letter is used, then no. of words = 5
 (ii) Two distinct letters are used, then no. of words

$${}^5C_2 \times \left(\frac{6!}{2!4!} \times 2 + \frac{6!}{3!3!} \right) = 10(30 + 20) = 500$$

- (iii) Three distinct letters are used, then no. of words

$${}^5C_3 \times \frac{6!}{2!2!2!} = 900$$

Total no. of words = 1405

- 25.** Let $S = \left\{ x : \cos^{-1} x = \pi + \sin^{-1} x + \sin^{-1} (2x+1) \right\}$.

Then $\sum_{x \in S} (2x-1)^2$ is equal to ____.

Ans. (5)

Sol. $\cos^{-1} x = \pi + \sin^{-1} x + \sin^{-1} (2x+1)$

$$2\cos^{-1} x - \sin^{-1} (2x+1) = \frac{3\pi}{2}$$

$$2\alpha - \beta = \frac{3\pi}{2} \text{ where } \cos^{-1} x = \alpha, \sin^{-1} (2x+1) = \beta$$

$$2\alpha = \frac{3\pi}{2} + \beta$$

$$\cos 2\alpha = \sin \beta$$

$$2\cos^2 \alpha - 1 = \sin \beta$$

$$2x^2 - 1 = 2x + 1$$

$$x^2 - x - 1 = 0$$

$$\Rightarrow n = \frac{1 \pm \sqrt{5}}{2} = \begin{cases} n = \frac{1 + \sqrt{5}}{2} \text{ rejected} \\ n = \frac{1 - \sqrt{5}}{2} \end{cases}$$

$$\therefore 4x^2 - 4x = 4$$

$$(2x-1)^2 = 5$$